

On combining significances. Some trivial examples.

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Abstract

For Poisson distribution $Pois(n, \lambda)$ with $\lambda \gg 1$, $n \gg 1$ we propose to determine significance as $S = \frac{n_{obs} - \lambda}{\sqrt{\lambda}}$. The significance S coincides up to sign with often used significance. For experiments which measure the same quantities the natural but not unique rule for significance combining is

$$S_{comb}(S_1, S_2) = \frac{S_1\sigma_1 + S_2\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}},$$

where σ_1 and σ_2 are variations. We also propose the rule for significances combining for the case with systematic errors.

“Suppose one experiment sees a 3-sigma effect and another sees a 4-sigma effect. What is combined significance?” [1].

In this note we discuss the problem of significances combining on the example of Poisson distribution¹. Namely for Poisson distribution $Pois(n, \lambda)$ with $\lambda \gg 1$, $n \gg 1$ we propose to determine significance as $S = \frac{n_{obs} - \lambda}{\sqrt{\lambda}}$. The significance S coincides up to sign with often used significance. For experiments which measure the same quantities the natural but not unique rule for significance combining is

$$S_{comb}(S_1, S_2) = \frac{S_1\sigma_1 + S_2\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}},$$

where σ_1 and σ_2 are variations. We also propose the rule for significances combining for systematic errors.

Suppose the CMS experiment detects in July 2010 10300 events with the expectation 10000 events and in august it detects 9700 events with the expectation 10000 events. The probability to detect n events is determined by Poisson formula

$$Pois(n, \lambda) = \frac{\lambda^n}{n!} e^{-\lambda}. \quad (1)$$

For $\lambda \gg 1$, $n \gg 1$ Poisson distribution is approximated by normal distribution with mean $\mu = \lambda$ and variance $\sigma = \sqrt{\lambda}$.

For $\lambda \gg 1$, $n_{obs} \gg 1$ the often used significance S is determined by the approximate formula

$$S = \frac{|n_{obs} - \lambda|}{\sqrt{\lambda}}. \quad (2)$$

For our example we find the july and august CMS significances

$$S_{july} = \frac{|10300 - 10000|}{\sqrt{10000}} = 3, \quad (3)$$

$$S_{august} = \frac{|9700 - 10000|}{\sqrt{10000}} = 3, \quad (4)$$

¹See also [2, 3]

If we deal with collection of CMS data for july plus august we find that data are described by Poisson distribution with $\lambda_{july+august} = \lambda_{july} + \lambda_{august}$ and $n_{obs,july+august} = n_{obs,july} + n_{obs,august}$. In accordance with the formula (2) we find that july + august significance is

$$S_{july+august} = \frac{|n_{obs,july} + n_{obs,august} - \lambda_{july} - \lambda_{august}|}{\sqrt{\lambda_{july} + \lambda_{august}}} = 0 \quad (5)$$

in perfect agreement with the theory expectations. This trivial example illustrates the fact that it is impossible to combine only significances, i.e.

$$S_{july+august} \neq F(S_{july}, S_{august}) \quad (6)$$

One of the possible solutions is to determine significance as

$$S = \frac{n_{obs} - \lambda}{\sqrt{\lambda}}. \quad (7)$$

The definition (7) coincides with often used significance definition up to sign. For the case of events excess it is positive and for the opposite case it is negative. For the definition (7) the proposed rule for significances combining is

$$S_{comb}(S_1, S_2) = \frac{S_1\sqrt{\lambda_1} + S_2\sqrt{\lambda_2}}{\sqrt{\lambda_1 + \lambda_2}} = \frac{S_1\sigma_1 + S_2\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}. \quad (8)$$

The rule (8) is in fact trivial generalization of Stouffer's method [4] and for Poisson statistics it looks rather natural². The generalization of the asymptotic formula (8) to the case of not big λ_1 and λ_2 is straightforward.

Other lesson from our example and formula (8) is that the significance combining not necessary leads to increase of the significance.

Note that the situation changes if we investigate the case when the parameter λ in Poisson distribution is not known. For our example we find that $\lambda_{july} = 10300$, $\lambda_{august} =$

²The rule (8) is the realization of the fact that the sum of Poisson processes with λ_1 and λ_2 is the Poisson process with $\lambda = \lambda_1 + \lambda_2$.

9700 and $\lambda_{july+august} = 20000$. In the assumption that $\lambda_{july} = \lambda_{august}$ the analysis of july+august data gives $\lambda_{july} = \lambda_{august} = 10000$

Other example is the case when CMS detects in july 10300 events with the expectation 10000 and in august it detects 10100 events with the expectation 10000.

Again we find

$$S_{july} = \frac{10300 - 10000}{\sqrt{10000}} = 3, \quad (9)$$

$$S_{august} = \frac{10100 - 10000}{\sqrt{10000}} = 1, \quad (10)$$

And for CMS data for july+august we find

$$S_{july+august} = \frac{400}{\sqrt{20000}} = \frac{4}{\sqrt{2}} \approx 2.8. \quad (11)$$

It is not difficult to take into account systematic effects related with nonexact knowledge of the parameter λ in Poisson distribution (1). Namely, suppose theoretical uncertainty in the λ parameter calculation is $\epsilon\lambda$. For such uncertainty the generalization of the formula (7) is

$$S = \frac{n_{obs} - \lambda}{\sqrt{\lambda + (\epsilon\lambda)^2}}. \quad (12)$$

The generalization of the formula (8) looks

$$S_{comb}(S_1, S_2) = \frac{S_1\sqrt{\lambda_1 + (\epsilon\lambda_1)^2} + S_2\sqrt{\lambda_2 + (\epsilon\lambda_2)^2}}{\sqrt{\lambda_1 + \lambda_2 + (\epsilon(\lambda_1 + \lambda_2))^2}} \quad (13)$$

Note that combining significances for experiments with different cuts in general does not help in search for new physics. Really suppose we look for the number of isolated muons with transverse momentum:

- (a). $100 \text{ GeV} < p_T < 200 \text{ GeV}$,
- (b). $p_T > 200 \text{ GeV}$,
- (c). $p_T > 100 \text{ GeV}$.

It is evident that the combination of the (a) “experiment” with the (b) “experiment” is equivalent to the (c) “experiment”. And it is not clear which experiment (a), (b) or (c) will give the biggest significance (the biggest evidence in favour of new physics). The details will depend on the tested model of new physics.

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References

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